Compressive Buckling of Sandwich Plates on Longitudinal Elastic Line Supports

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In this paper an analysis of the stiffness requirements for compressive buckling of simply supported rectangular plates supported on longitudinal lines of elastic springs with stiffness independent of buckle wave length is extended to honeycomb core sandwich plates with isotropic faces and with cores which carry no bending stress and have orthotropic transverse shearing properties. The axes of orthotropy of the core are parallel to the plate edges. Calculations have been carried out for long plates with various core stiffness properties. The results indicate that the elastic support stiffness required to achieve buckling with nodes at the supports increases significantly as the core shear stiffness decreases. When the support stiffness depends on the buckle wave-length, however, as for a longitudinal stiffener, the required beam bending stiffness of the stiffener is not affected as much and may actually decrease with decreasing core shear stiffness.

Nomenclature

a_n, b_n, c_n = buckle pattern coefficients
b = distance between support lines
D = plate bending stiffness
EI_{eff} = effective bending stiffness of stiffener
j, n = integers
k_x = critical load parameter $(N_x b^2/\pi^2 D)$
L = plate length
N = number of plate bays
N_x = critical load per unit width
P = stiffener load
P_{cr} = Euler buckling load of stiffener $(\pi^2 EI_{\text{eff}}/L^2)$
r_x, r_y = core shearing stiffness parameters $(s_x b^2/\pi^2 D,\ S_y b^2/\pi^2 D)$
S_x, S_y = transverse shear stiffness of plate in x-z and y-z planes
w = transverse deflection of plate
β = buckle aspect ration (λ/b)
γ = support stiffness parameter $(\gamma b^3/\pi^4 D)$
γ_x, γ_y = transverse shear strains
λ = half wavelength of plate buckle
ψ = line support stiffness

I. Introduction

IN many structural configurations for which the possibility of buckling in a number of different modes exists, an optimum or near-optimum design is often obtained when the elements are proportioned so that the buckling loads for the various buckle patterns are identical. In particular, plates with longitudinal or transverse stiffeners may be designed so that the compressive buckling stress for general instability is equal to that for buckling with nodes at the stiffeners. In the course of an investigation of the efficiency of various types of construction for a fighter wing structure, the critical support stiffness for a stiffened honeycomb sandwich was desired. The results of the ensuing study indicated that the required support stiffness increased significantly as the core shear stiffness decreased. This phenomenon can be attributed to the fact that the critical load for local buckling with nodes at the supports approaches the critical general instability load for the plate with very stiff supports as the core shear stiffness decreases. A more detailed numerical investigation shows that the support stiffness can be decreased from these large values with only a slight decrease in critical stress. In addition, when the support stiffness is supplied by longitudinal stiffeners, the required beam bending stiffness is not affected as drastically and may actually decrease with decreasing core shear stiffness. The results of the studies are described in the present paper.

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II. Analysis

An analysis of the stiffness requirements for compressive buckling of plates supported on longitudinal lines of elastic springs is given in Ref. 1. In the present report, the analysis is extended to sandwich plates with isotropic faces and axes of orthotropy of the core parallel to the plate edges and which rest on equally spaced longitudinal elastic springs of equal stiffness (Fig. 1). The potential energy of the plate-spring system is given by Ref. 2 as

$$U = \frac{1}{2}\int_0^{Nb}\int_0^{\lambda} < D\left\{\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \gamma_x\right)\right]^2 + 2\nu\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \gamma_x\right)\right] \times \left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} - \gamma_y\right)\right] + \left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} - \gamma_y\right)\right]^2 + \frac{1-\nu}{2} \times \left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \gamma_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} - \gamma_x\right)\right]^2\right\} + S_x\gamma_x{}^2 + S_y\gamma_y{}^2 - N_x\left(\frac{\partial w}{\partial x}\right)^2 > dx\,dy + \tfrac{1}{2}\psi\sum_{j-1}^{N-1}\int_0^{\lambda}(w_{y=jb})^2\,dx \qquad (1)$$

where the longitudinal integration is carried out over one half-wave of the repetitive buckle pattern. Let the assumed buckle pattern be represented by

$$w = \sin\frac{\pi x}{\lambda}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{Nb} \qquad (2a)$$

$$\gamma_x = \cos\frac{\pi x}{\lambda}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{Nb} \qquad (2b)$$

$$\gamma_y = \sin\frac{\pi x}{\lambda}\sum_{n=1}^{\infty} c_n \cos\frac{n\pi y}{Nb} \qquad (2c)$$

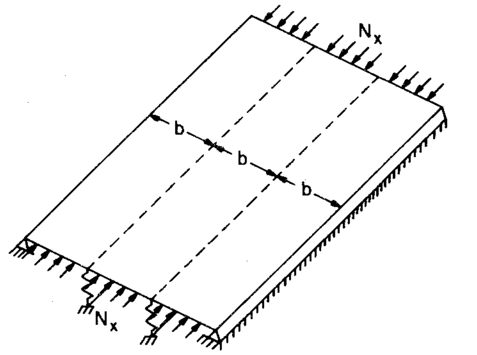

Fig. 1 Simply supported plate on elastic line supports.

which satisfy edge conditions of no transverse displacement, no moment, and no rotation in planes parallel to the edge of line elements normal to the plane of the plate. The substitution of Eqs. (2) into Eq. (1) then yields

$$U = \frac{\lambda N b}{8}\sum_{n=1}^{\infty} < D\left\{\left[\frac{\pi^2}{\lambda^2} + \frac{1-v}{2}\left(\frac{n\pi}{Nb}\right)^2\right]\left(\frac{\pi}{\lambda}a_n - b_n\right)^2 + (1+v)\frac{n\pi}{Nb}\frac{\pi}{\lambda}\left(\frac{\pi}{\lambda}a_n - b_n\right)\left(\frac{n}{Nb}a_n - c_n\right) + \left[\left(\frac{n\pi}{Nb}\right)^2 + \frac{1-v}{2}\frac{\pi^2}{\lambda^2}\right]\left(\frac{n\pi}{Nb}a_n - c_n\right)^2\right\} + S_x b_n^2 + S_y c_n^2 - N_x\left(\frac{\pi}{\lambda}a_n\right)^2 > + \frac{\psi\pi}{4}\sum_{j=1}^{N-1}\left(\sum_{n=1}^{\infty} a_n \sin\frac{n\pi j}{N}\right)^2 \qquad (3)$$

For equilibrium to exist in the buckled shape, the following conditions must hold

$$\frac{\partial U}{\partial a_n} = \frac{\partial U}{\partial b_n} = \frac{\partial U}{\partial c_n} = 0 \qquad (n = 1,2,\ldots\infty) \qquad (4)$$

which yield

$$\frac{b_n}{\pi a_n/b} = \frac{\frac{1}{\beta}\left(1+\frac{n^2}{N^2}\beta^2\right)\left[\frac{1-v}{2}\left(1+\frac{n^2}{N^2}\beta^2\right) + r_y\beta^2\right]}{\left(1+\frac{1-v}{2}\frac{n^2}{N^2}\beta^2 + r_x\beta^2\right)\left(\frac{1-v}{2}+\frac{n^2}{N^2}\beta^2 + r_y\beta^2\right) - \left(\frac{1+v}{2}\right)^2\frac{n^2}{N^2}\beta^2} \qquad (5a)$$

$$\frac{c_n}{\pi a_n/b} = \frac{\frac{n}{N}\left(1+\frac{n^2}{N^2}\beta^2\right)\left[\frac{1-v}{2}\left(1+\frac{n^2}{N^2}\beta^2\right) + r_x\beta^2\right]}{\left(1+\frac{1-v}{2}\frac{n^2}{N^2}\beta^2 + r_x\beta^2\right)\left(\frac{1-v}{2}+\frac{n^2}{N^2}\beta^2 + r_y\beta^2\right) - \left(\frac{1+v}{2}\right)^2\frac{n^2}{N^2}\beta^2} \qquad (5b)$$

$$0 = a_n + \gamma\beta^2\sum_{m=1}^{\infty} a_n\left(\frac{2}{N}\sum_{j=1}^{N-1}\sin\frac{n\pi j}{N}\sin\frac{m\pi j}{N}\right) \frac{\left[r_x r_y\beta^2 + \frac{1-v}{2}\left(r_x + r_y\frac{n^2}{N^2}\beta^2\right)\right]\left(1+\frac{n^2}{N^2}\beta^2\right)^2}{\left(1+\frac{1-v}{2}\frac{n^2}{N^2}\beta^2 + r_x\beta^2\right)\left(\frac{1-v}{2}+\frac{n^2}{N^2}\beta^2 + r_y\beta^2\right) - \left(\frac{1+v}{2}\right)^2\frac{n^2}{N^2}\beta^2} - k_x$$

$$n = 1,2,\ldots \qquad (5c)$$

Equations (5c) are similar to Eq. (A4) of Ref. 3 and can be solved as in that paper for the stability criterion

$$1+\gamma\beta^2\left(\sum_{s=0}^{\infty} R_{2s+q/N} + \sum_{s=1}^{\infty} R_{2s-q/N}\right) = 0$$

$$q = 1,2,\ldots N-1 \qquad (6a)$$

with

$$R_{2s\pm q/N} = -k_x + \frac{\left[1+\left(2s\pm\frac{q}{N}\right)^2\beta^2\right]^2\left\{r_x r_y\beta^2 + \frac{1-v}{2}\left[r_x + \left(2s\pm\frac{q}{N}\right)^2 r_y\beta^2\right]\right\}}{\left[1+\frac{1-v}{2}\left(2s\pm\frac{q}{N}\right)^2\beta^2 + r_x\beta^2\right]\left[\frac{1-v}{2}+\left(2s\pm\frac{q}{N}\right)^2\beta^2 + r_y\beta^2\right] - \left(\frac{1-v}{2}\right)^2\left(2s\pm\frac{q}{N}\right)^2\beta^2} \qquad (6b)$$

and

$$k_x = \frac{\left[r_x r_y\beta^2 + \frac{1-v}{2}(r_x + r_y\beta^2)\right](1+\beta^2)^2}{\left(1+\frac{1-v}{2}\beta^2 + r_x\beta^2\right)\left(\frac{1-v}{2}+\beta^2 + r_y\beta^2\right) - \left(\frac{1+v}{2}\right)^2\beta^2} \qquad (6c)$$

whichever yields the lowest buckling load. For a long plate, the buckling load must be minimized with respect to the half-wave length parameter β as well.

In the case of a plate with an isotropic core ($r_x = r_y = r$) the stability criterions given by Eqs. (6a) and (6b) can readily be put into closed form. After much manipulation and the use of Eq. (6.495.2) of Ref. 4, we have

$$\gamma = \frac{8}{\pi^2}r \times \left\{\frac{\left[\left(1+\frac{4r^2\beta^2}{k_x}\right)^{1/4} + \left(1+\frac{4r^2\beta^2}{k_x}\right)^{-1/4}\right]^2}{\pi(\alpha_2)^{1/2}}\frac{\sin\pi(\alpha_2)^{1/2}}{\cos\frac{\pi q}{N} - \cos\pi(\alpha_2)^{1/2}} + \frac{\left[\left(1+\frac{4r^2\beta^2}{k_x}\right)^{1/4} - \left(1+\frac{4r^2\beta^2}{k_x}\right)^{-1/4}\right]^2}{\pi(\alpha_1)^{1/2}} \times \frac{\sin h\,\pi(\alpha_1)^{1/2}}{\cos h\,\pi(\alpha_1)^{1/2} - \cos\frac{\pi q}{N}}\right\}^{-1}$$

$$q = 1,2,\ldots,N-1 \qquad (7a)$$

with

$$\left.\begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right\} = \frac{k_x}{2\beta^2 r}\left[\left(1+\frac{4r^2\beta^2}{k_x}\right)^{1/2} \pm \left(2\frac{r}{k_x} - 1\right)\right] \qquad (7b)$$

Maximum Load-Carrying Capacity for Long Plates

The value of β^2 which minimizes Eq. (6c) is given by the real positive solution, if any, of the cubic equation

$$r_y(r_y+1)\left(\frac{1-v}{2}+r_x\right)^2(\beta^2)^3 - r_y\left(\frac{1-v}{2}+r_x\right) \times \left(r_x r_y - \frac{3+v}{2}r_y + v r_x - 3\frac{1-v}{2}\right)(\beta^2)^2 - \frac{1-v}{2} \times \left[2r_y r_x^2 + \frac{3+v}{2}r_x^2 - (3+v)r_x r_y - 3\frac{1-v}{2}r_y\right]\beta^2 - \left(\frac{1-v}{2}\right)^2(r_x^2 - r_y) = 0 \qquad (8)$$

If $r_x = r_y = r$ (an isotropic sandwich plate) the solution of Eq. (11) is

$$\beta^2 = (r - 1/r + 1) \quad (r \geqq 1) \qquad (9a)$$

and the buckling load is

$$k_x = (2r/1+r)^2 \qquad (9b)$$

When r is less than unity, β^2 is put equal to zero in Eq. (9c) and the buckling load becomes

$$k_x = r \tag{9c}$$

Another case which is readily obtained is that for which $r_y = \infty$, $r_x = r$. Then

$$\beta^2 = \left(1 - \frac{3+\nu}{2} \Big/ r + \frac{1-\nu}{2}\right) \quad \text{if} \quad r \geqq \frac{3+\nu}{2} \tag{10a}$$

in which case

$$k_x = 4\left(r - \frac{1+\nu}{2} \Big/ r + \frac{1-\nu}{2}\right) \tag{10b}$$

If r is less than $(3+\nu/2)$, β^2 is put equal to zero and

$$k_x = r + (1-\nu/2) \tag{10c}$$

For other cases Eqs. (8) must be solved and the result substituted into Eq. (6c).

Required Support Stiffness for Long Plates

The support stiffness required for the sandwich plate to achieve its maximum load-carrying capacity is obtained by putting the value of k_x obtained from the preceding section into Eq. (6b) and computing the value of γ from Eq. (6a) as a function of β. The required values of γ, is the maximum value of the curve so obtained. In the case of a sandwich plate with an isotropic core, for example, we have from Eqs. (7a), (7b), and (9b)

$$\gamma = \frac{8}{\pi^2} r \times \left\langle \frac{\{[1+(r+1)^2\beta^2]^{1/4} - [1+(r+1)^2\beta]^{-1/4}\}^2}{\pi(\alpha_1)^{1/2}} \times \frac{\sinh \pi(\alpha_1)^{1/2}}{\cosh \pi(\alpha_1)^{1/2} - \cos \pi \frac{q}{N}} + \frac{\{[1+(r+1)^2\beta^2]^{1/4} + [1+(r+1)^2\beta^2]^{-1/4}\}^2}{\pi(\alpha_2)^{1/2}} \times \frac{\sin \pi(\alpha_2)^{1/2}}{\cos \pi \frac{q}{N} - \cos \pi(\alpha_2)^{1/2}} \right\rangle^{-1}$$

$$q = 1, 2, \ldots, N-1 \tag{11a}$$

with

$$\left.\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix}\right\} = \frac{2r[1+(r+1)^2\beta^2]^{1/2} \pm (r^2+1)}{(r+1)^2\beta^2} \tag{11b}$$

In general, the value of γ for which $q = 1$ can be expected to govern.

III. Results and Discussion

Calculations were made of the minimum support stiffness required to produce a buckling mode with nodes at the supports for long plates. The number of bays N was taken equal to infinity since the results were not expected to vary significantly for plates with three or more bays. The core shear stiffness parameter r_y was taken equal to $0.4r_x$, $1.0r_x$, and $2.5r_x$.

For given values of r_x and with k_x determined from Eq. (9), the buckle aspect ratio β was varied until a maximum value of the support stiffness parameter γ was obtained. Some typical variations of γ with β are shown in Fig. 2. The maximum values of γ obtained from these calculations are shown in Fig. 3.

It will be noted that in all cases the required support stiffness increases rapidly as the core shear stiffness decreases. Curves of the variation of buckling load with support stiffness were obtained for a long plate with many bays and with various values of the core shear stiffness parameter. Again the support stiffness parameter was maximized with respect to buckle wavelength for given buckling load. Because of the large number of calculations, results were obtained only for the case of an isotropic core ($S_x = S_y$). These are shown in Fig. 4 where it is seen that

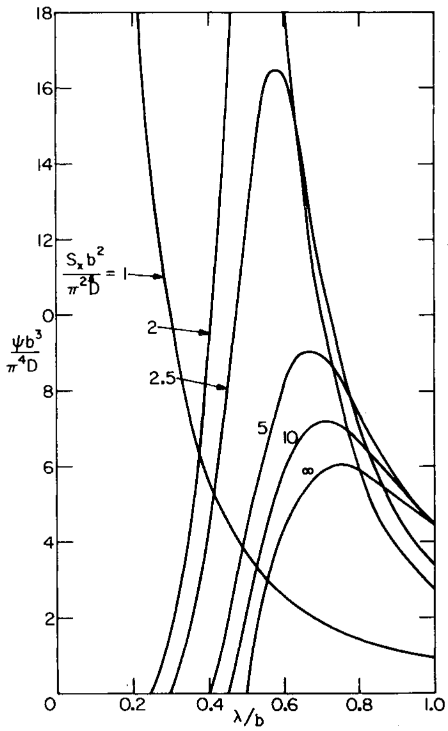

Fig. 2 Variation with buckle wavelength of support stiffness required to achieve nodes at the supports ($N = \infty$, $S_x = S_y$).

as the core shear stiffness decreases the buckling load becomes increasingly insensitive to support stiffness. Thus a drastic decrease in support stiffness can be allowed with only a corresponding minor decrease in allowable load.

Since these results apply only to plates with line supports having spring stiffnesses independent of buckle wavelength, another study was carried out for an isotropic sandwich plate ($S_x = S_y$) for which the support stiffness is provided by identical longitudinal stiffeners. Since calculations were available for a plate with an infinite number of stiffeners only, the length of

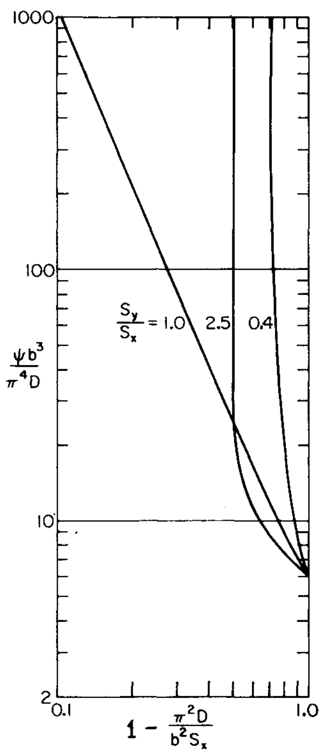

Fig. 3 Values of spring stiffness required for buckling of long plates with nodes at the supports ($\nu = 0.3$, $N = \infty$).

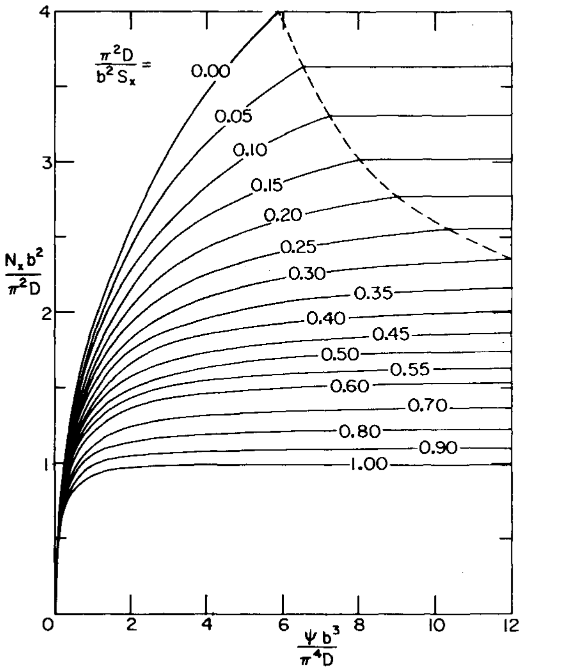

Fig. 4 Load-support stiffness curves for long plates ($N = \infty$, $S_x = S_y$).

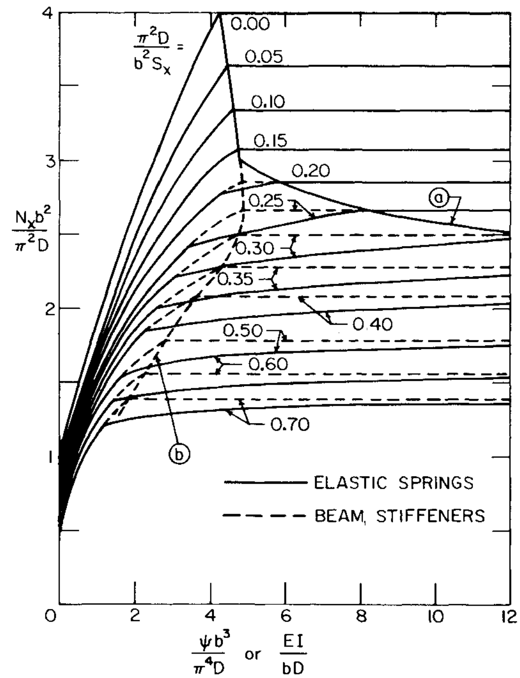

Fig. 5 Load-support stiffness curves for a plate of finite length on either longitudinal spring supports or stiffeners ($L/b = 1$, $S_x = S_y$, $N = \infty$). ⓐ= spring stiffness parameter for nodes at the supports; ⓑ= beam stiffness parameter for nodes at the supports.

the plate was made finite; otherwise the stiffeners would be ineffective. The stiffness parameter for such stiffeners is shown in Ref. 1 to be given by

$$\frac{\psi b^3}{\pi^4 D} = \frac{1}{\beta^4}\frac{EI_{\text{eff}}}{bD}\left(1 - \frac{P}{P_{\text{cr}}}\frac{\beta^2}{L^2/b^2}\right) \tag{12}$$

For convenience, the length L of the plate was taken equal to the distance b between stiffeners. The stiffeners were assumed to carry little axial load so that P could be neglected. Finally, the neutral axis of the stiffeners was assumed to coincide with the plate middle surface so that the effective stiffener bending stiffness could be considered to be independent of buckle wavelength. The curves of Fig. 5 show both the constant spring support stiffness parameter and the constant stiffener bending stiffness parameter required to prevent buckling at a given load for various values of the core shear stiffness parameter. These curves were obtained by determining maximum values of both of the stiffness parameters from the values obtained from Eq. (7) for equal to 1, 1/2, 1/3, 1/4,... corresponding to buckle half-wave lengths equal to those fractions of the plate length.

If the plate buckles in a single longitudinal half-wave, the two stiffness parameters are equal in the present case. The curves of Fig. 5 indicate that this is so for the lower values of core shear flexibility or of buckling load. As the core shear stiffness decreases, however, the plate on spring supports tends to buckle in larger numbers of longitudinal half-waves requiring increasingly larger support stiffness, whereas the stiffened plate continues to buckle in one half wave with the required bending stiffness changing little. Whereas the required spring support stiffness for nodes at the supports increases rapidly as the core shear stiffness decreases, the required stiffener bending stiffness only increases slightly at first and then decreases significantly. It would appear, then, that the effective spring stiffness of a stiffener increases much more rapidly with decreasing buckle aspect ratio than is required for stability of the structure.

References

[1] Anderson, R. A. and Semonian, J. W. "Charts Relating the Compressive Buckling Stress of the Longitudinally Supported Plates to the Effective Deflectional and Rotational Stiffness of the Supports," Rept. 1202, 1954, NACA.

[2] Plantema, F. J., "Sandwich Construction," *The Bending and Buckling of Sandwich Beams, Plates, and Shells,* Wiley, New York, 1966, pp. 63–65.

[3] Seide, P. and Stein, M., "Compressive Buckling of Simply Supported Plates with Longitudinal Stiffeners," TN 1825, 1949, NACA.

[4] Adams, E. P. and Hippisley, R. L., *Smithsonian Mathematical Formulae and Tables of Elliptic Functions,* Second Reprint, Smithsonian Misc. Coll., Vol. 74, No. 1, 1947.